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IDENTITIES AND LINEAR DEPENDENCE

BY

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ABSTRACT

Central polynomial identities are used to construct alternating central identities by which new identities are obtained. These identities express the linear dependence of n^2+1 generic matrices, and so yield slight generalizations and simplified proofs of a result of Formanek, the theorem about Azumaya algebras of M. Artin and a recent result of Cauchon.

Among the many central identities of a matrix ring we introduce new identities $\delta[x_1,\cdots,x_{n^2},y_1,\cdots,y_m]$ which are alternating, homogeneous and linear only in the x_i, which can be obtained from any regular central identity. These new types of identities yield a linear dependence $\Sigma(-1)^{i-1}\cdot\delta[x_1,\cdots,\hat{x}_i,\cdots,x_{n^2};y_1,\cdots,y_m]x_i=0$ which holds in every prime and semi-prime ring of "identity degree n". This linear dependence yields rather easily a result of Formanek [5] on the embedding of a prime PI-ring into a free C-module. Moreover, the generalization of this result when combined with an idea of Rowen [8] on the behavior of regular central identities yields a direct relatively simple proof fo the famous M. Artin-Procesi theorem ([3], [6]) on Azumaya algebras. The present proof differs from all other known proofs ([1], [3], [6], [8]) as it does not use any trace arguments and no reduction to prime rings or to homomorphic images of a generic matrix ring. In fact it enables us to prove that if R satisfies a certain identity $d_{n^2+1}[x;y]=0$ and has a regular central identity which does not vanish on all simple images R/M, then R is an Azumaya algebra, which is far less than the assumption that all identities of $M_n(Z)$ hold in R. Finally, this identity is used to prove along parallel lines the result of Formanek that a semi-prime PI-ring R with a noetherian center is C-noetherian, and a recent result of Cauchon ([9]) that ACC on two-sided ideals implies that R is noetherian.

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1. The polynomials $d_m[x;y]$

As a basic polynomial for generating identities of a matrix ring $M_n(K)$, we shall use the following polynomials:

$$d_m[x_1,\cdots,x_m;y_1,\cdots,y_{m-1}] = \Sigma \operatorname{sg}\sigma x_{\sigma(1)}y_1x_{\sigma(2)}\cdots x_{\sigma(m-1)}y_{m-1}x_{\sigma(m)} \tag{1}$$

where σ ranges over all permutations of $\{1,2,\cdots,m\}$. Our first result is:

THEOREM 1a. *If K is a commutative ring with a unit, then $M_n(K)$ satisfies $d_m[x,y]=0$ for all $m \geq n^2+1$, and for every $i \neq k$ there is a substitution for x's and y's so that $d_{n^2}[x;y]=c_{ik}$, where c_{ik} are the matrix units.*

PROOF. The polynomials $d_m[x;y]$ satisfy a recursive relation like determinants, that is:

$$\begin{aligned} &d_{m+1}[x_1,\cdots,x_{m+1};y_1,\cdots,y_m] \\ &= \sum_{i=1}^{m+1}(-1)^{i-1}d_m[x_1,\cdots,\hat{x}_i,\cdots,x_{m+1};y_1,\cdots,y_{m-1}]\,y_mx_i. \end{aligned} \tag{2}$$

Thus if $d_m[x;y]=0$ in $M_n(K)$ for $m=m_0$ then it will hold also for $m \geq m_0$.

Since $d_m[x;y]$ is multilinear and homogeneous it suffices to show that $d_m[x;y]=0$ for substitutions of the indeterminates by elements taken from a linear base of $M_n(K)$, but as such we have only n^2 different elements and so if $m \geq n^2+1$, at least two will be equal and hence $d_m[x;y]=0$, since $d_m[x;y]=0$ if two of the x_i's are equal.

To prove that $d_{n^2}[x;y]=c_{ik}$, we restrict ourselves to $(i,k)=(1,n)$ and to this end we substitute for the set $\{x_1,\cdots,x_{n^2}\}$ the ordered set $\{c_{11},c_{12},\cdots,c_{1n};c_{21},\cdots,c_{2n},\cdots,c_{n1},\cdots,c_{nn}\}$ and for $\{y_1,\cdots,y_{n^2-1}\}$ the set of $\{c_{11},c_{21},\cdots,c_{n-11},c_{n2},c_{12},\cdots,c_{1n},\cdots,c_{n-1n}\}$. Namely, the y's are chosen so that the monomial $x_1y_1x_2y_2\cdots y_{n^2-1}x_{n^2}$ will be equal to c_{1n}. In fact, this is the only monomial which will yield a nonzero element, since if $y_ix_\lambda y_{i+1} \neq 0$ then x_λ is uniquely determined, and therefore if $x_{\sigma(1)}y_1x_{\sigma(2)}\cdots x_{\sigma(n^2-1)}y_{n^2-1}x_{\sigma(n^2)}$ yields a nonzero element then $x_{\sigma(i)}=x_i$ for $i=2,\cdots,n^2-1$. Moreover, since $y_1=c_{11}$ we must have $x_{\sigma(1)}=c_{\alpha 1}$ and the only one of this form available is c_{11} which is equal to x_1. Thus $\sigma(1)=1$, and similarly $x_{\sigma(n)}=c_{nn}=x_n$. Thus $\sigma=$ identity and this completes the proof that $d_{n^2}(x;y]=c_{1n}$.

If R is a prime ring (an algebra over a commutative ring K) which satisfies a polynomial identity then its ring of quotients $Q(R)$ is a central simple algebra of dimension n^2 over its center, and we define $\operatorname{pid}(R)=n$ (-polynomial identity degree). If R is a semi-prime ring with a polynomial identity then we set $\mathrm{pid}(R) = \max \mathrm{pid}(R/P)$ where P ranges over all prime ideals P of R. Clearly, this generalizes the definition of pid for prime rings.

We can extend Theorem 1a to semi-prime rings.

THEOREM 1b: *If R is a semi-prime K-algebra and* $\mathrm{pid}(R) = n$ *then R satisfies $d_m[x; y] = 0$ for $m \geqq n^2 + 1$; and $d_{n^2}[x; y] \neq 0$ in R.*

PROOF: Consider first the case of a prime ring R; then R and its ring of quotients $Q(R)$ satisfy the same identities. If the center C of $Q(R)$ is finite then $R = Q(R) = M_n(C)$ and so Theorem 1a coincides with our case. If C is infinite, choose splitting field F of $Q(R)$. That is, $Q(R) \otimes F = M_n(F)$ and $M_n(F)$, $Q(R)$ have the same identities; hence the fact that $d_m[x; y] = 0$ in $M_n(F)$ for $m \geq n^2 + 1$ implies that it holds also in $Q(R)$ as well as in R. Also $d_{n^2}[x; y] \neq 0$ in $M_n(F)$ and since it is homogeneous and multilinear and $Q(R) = RC$, $Q(R) \otimes F = M_n(F)$, the polynomial $d_{n^2}[x; y]$ cannot vanish on R.

If R is semi-prime then R is a subdirect sum of the prime rings R/P which satisfy $d_m[x; y] = 0$ for $m \geqq n^2 + 1$, so it holds also in R. At least one of R/P has $\mathrm{pid}(R/P) = n$, and there $d_{n^2}[x; y] \neq 0$; hence this polynomial does not vanish in R.

2. Alternating central identities

A polynomial $p[x_1, \cdots, x_m, y_1, \cdots, y_t]$ is said to be an alternating polynomial (in the x's) if $p[x; y]$ is a homogeneous polynomial multilinear only in $x_1, \cdots, x_m$ and vanishes if two of the x_i are made equal. The following follows readily:

LEMMA 2. *A polynomial $p[x_1, \cdots, x_m, y_1, \cdots, y_t]$ is alternating in the x's if and only if it has the form:*

$$p[x; y] = \sum_{(\tau)} p_{\tau_0} d_m[x_1, \cdots, x_m; p_{\tau_1}, \cdots, p_{\tau_{m-1}}] p_{\tau_m} \tag{3}$$

where the p_{τ_i} are polynomials in the y's only.

Indeed, write $p[x, y] = \Sigma_{\rho,\sigma} \alpha_{\rho\sigma} l_{\rho 0} x_{\sigma(1)} l_{\rho 1} x_{\sigma(2)} \cdots l_{\rho m-1} x_{\sigma(m)} l_{\rho m}$ where l_{ρ_i} are monomials in the y's and the $\alpha_{\rho\sigma}$ are scalar (of K). Since it is multilinear in the x's and alternating, it follows readily that the interchanging of two x_i will yield a change of sign of $p[x; y]$, and this readily implies that $\alpha_{\rho\sigma} = \alpha_{\rho 1} \mathrm{sg}\, \sigma$ where $\mathrm{sg}\sigma$ is the sign of the respective permutation. Summing up $\Sigma_{\rho\sigma} = \Sigma_\rho \Sigma_\sigma$ clearly yields (3). The converse is evident.

A polynomial $\delta[x;y]$ will be called an *alternating central identity* of a ring R, if $\delta[x;y]$ is alternating in the x's and it is a central identity for R, that is, $\delta[x;y]$ is the center of R for all values of x and y but it is not identically zero.

THEOREM 3. *Let R be a semi-prime ring with* $\mathrm{pid}(R)=n$, *then R has an alternating central identity* $\delta[x_1,\cdots,x_{n^2};y_1,\cdots,y_m]$, *which holds also for all* $M_n(K)$, *K an arbitrary commutative ring; and for each alternating identity the relation*

$$\sum_{i=1}^{n^2+1}(-1)^{i-1}\delta[x_1,\cdots,\hat{x}_i,\cdots,x_{n^2+1};y_1,\cdots,y_m]x_i=0 \tag{4}$$

holds in R. ($\hat{x}_i$ denotes the omitting of the x_i from the variables of $\delta[x;y]$).

PROOF. Let $q[u_0,u_1,\cdots,u_t]$ be any central identity for $M_n(K)$ which holds for every commutative ring K (with a unit) and which is homogeneous and linear in at least one of the indeterminates, say u_0. (These were called *regular* central identities in [8].) Such a central identity exists, e.g. the Formanek or Razmyslov polynomials ([4],[7]). Consider the polynomial:

$$\delta[x;y]=q[d_{n^2}[x_1,\cdots,x_{n^2};y_1,\cdots,y_{n^2-1}]y_{n^2};y_{n^2+1},\cdots,y_m]$$

where $m=n^2+t$. $\delta[x;y]$ is obtained from $q[u]$ by the replacements $u_0=d_{n^2}[x_1,\cdots,x_{n^2};y_1,\cdots,y_{n^2-1}]y_{n^2}$ and $u_j=y_{n^2+j}$ for $j\geqq 1$. Since $q[u]$ is homogeneous and linear in u_0, $\delta[x;y]$ will be alternating in the x's. To prove that it is a central identity we consider first the case where R is a prime ring. Again if the center of the ring of quotients $Q(R)$ is finite then $R=Q(R)=M_n(C)$; since $q[a_0,a_1,\cdots,a_t]\neq 0$ for some $a_i\in R$ and is linear in u_0, then $q[c_{ik},a_1,\cdots,a_t]\neq 0$ for some matrix unit c_{ik}. It follows, by Theorem 1a, that $d_{n^2}[x;y]=c_{ij}$ for $j\neq i$ for some substitution in $M_n(C)$, and thus if we set $y_{n^2}=c_{ik}$ and $y_{n^2+\lambda}=a_\lambda$ we get $\delta[x;y]=q[c_{ik},a_1,\cdots,a_t]\neq 0$. The same proof shows that $\delta[x;y]$ does not vanish in any matrix ring $M_n(K)$ over any commutative ring K. Hence, if the center of $Q(R)$ is an infinite field, R and $Q(R)=RC$ and $Q(R)\otimes F\cong M_n(F)$ will have the same identities and in particular $\delta[x;y]$ which does not vanish in $M_n(F)$ will not vanish in R. On the other hand it gets only central values in $M_n(F)$ and hence it is a central alternating identity for R.

Next, if R is semi-prime, then R is a subdirect sum of the prime rings R/P and, at least for one P, $\mathrm{pid}(R/P)=n$. Now, $\delta[x;y]$ is a central identity for R/P of pid $=n$, and if $\mathrm{pid}(R/P)<n$ then $\delta[x;y]=0$ in R/P since central identities of $M_n(K)$ are identities for $M_m(K)$ for $m<n$. Thus $\delta[x,y]$ is a central identity for R since it does not vanish for at least one R/P, and this completes the proof of the first part of our theorem.

Let $p[x_1,\cdots,x_m;y_1,\cdots,y_t]$ be any alternating polynomial (not necessarily central); then it follows by (3) and (2) that:

$$\sum_{i=1}^{m+1}(-)^{i-1}p[x_1,\cdots,\hat{x}_i,\cdots,x_{m+1};y_1,\cdots,y_t]x_i$$
$$=\sum_{i=1}^{m+1}\sum_{\tau}(-1)^{i-1}p_{\tau_0}d_m[x_1,\cdots,\hat{x}_i,\cdots,x_{m+1};p_{\tau_1},\cdots,p_{\tau_{m-1}}]p_{\tau_m}x_i$$
$$=\sum_{\tau}p_{\tau_0}d_{m+1}[x_1,x_2,\cdots,x_{m+1};p_{\tau_1},\cdots,p_{\tau_{m-1}},p_{\tau_m}].$$

In particular this implies a more general result than (4):

COROLLARY 4. *If R is any ring satisfying the identity $d_{n^2+1}[x;y]=0$ (e.g. R is a semi-prime ring with* pid$(R)=n$*), and $p[x_1,\cdots,x_{n^2};y_1,\cdots,y_t]$ is any alternating polynomial in n^2 indeterminates $x_1,\cdots,x_n$ then R satisfies the identity*:

$$\sum_{i=1}^{n^2+1}(-1)^{i-1}p[x_1,\cdots,\hat{x}_i,\cdots,x_{n^2+1};y_1,\cdots,y_t]x_i=0. \tag{5}$$

Indeed, R satisfies the identity $d_{n^2+1}[x;y]=0$ by Theorem 1b, and thus (5) follows by the relation proved above for $m=n^2$.

Finally (5) is the required relation (4) of our theorem.

REMARK. Note that (4) yields the linear dependence of n^2+1 generic matrices in the ring generated by these matrices.

The proof of Theorem 3 actually proves the more general case:

COROLLARY 5. *If R is a K-algebra which satisfies $d_{n^2+1}[x;y]=0$ and R has a regular central identity $q[u]$ which does not vanish on all matrix rings $M_n(\bar{K})$, $\bar{K}$ homomorphic image of K, and if either* pid$(R)=n$ *or $q[u]$ attains non nil elements, then R has an alternating central identity $\delta[x_1,\cdots,x_{n^2};y_1,\cdots,y_m]$ with the same property and* (4) *holds in R.*

In the proof of Theorem 3, the existence of a regular central identity $q[u]$ was known for prime rings and hence for semi-prime rings with pid$(R)=n$. In the present case, we again choose $\delta[x;y]=q[d_{n^2}[x;y]y_{n^2},y_{n^2+1},\cdots,y_m]$ and it remains only to show that $\delta[x;y]\neq 0$ in R. Indeed, since $d_{n^2+1}[x;y]=0$ holds also in every R/P, P a prime ideal in R, it follows that pid$(R/P)\leq n$. If pid$(R/P)=n$ for some P then we can reproduce the proof of Theorem 3. Otherwise, $q[a]\in\cap P=$ Lower radical, and so $q[a]$ is nil, which is impossible in our case.

3. Application I

We apply (4) to obtain a refinement of a result of Formanek [5]; but first we introduce the following:

Let R be a ring with a center C and assume that R has an alternating central identity $\delta[x;y] = \delta[x_1,\cdots,x_m;y_1,\cdots,y_t]$. Denote by Δ the additive group generated by all values $\{\delta[a;b]\}$, $a_j, b_j \in R$; then Δ is an ideal in C and by definition $\Delta \neq 0$. Indeed for every $c \in C$, $\delta[a_1,\cdots,a_m;b_1,\cdots]c = \delta[a_1c, a_2\cdots]$ since $\delta[x;y]$ is linear and homogeneous in x_1, and by definition Δ is an additive subgroup of C.

THEOREM 6. *If R satisfies $d_{m+1}[x;y]=0$ and $\delta[x_1,\cdots,x_m;y_1,\cdots]$ is an alternating central identity for R, (e.g. if R is semi-prime and* pid$(R)=n$ *then such identities exist for $m = n^2$), then*

1) *for every ideal A of R, $\Delta A \subseteq (A \cap C)R \subseteq A$;*

2) *the ring of quotients R_δ is a free C_δ-module of rank m for every $\delta \in \Delta$ for which $R_\delta \neq 0$ (i.e. $\delta^h R \neq 0$ for all $h \geq 0$);*

3) *If, for some $\delta \in \Delta$, δ is a nonzero divisor in C then there exists a subalgebra $A = \Sigma_{i=1}^m Ca_i$ which is free of rank m over C and for which $\delta^2 R \subseteq A \subseteq R$; moreover, R can be embedded in a free C-module of rank m.*

Indeed, consider the coefficients of (4) and note that if $a \in A$, then $\delta[r_1,\cdots,r_{i-1},a,r_{i+1},\cdots,r_m;s_1,\cdots] \in A \cap C$. Consequently, for $a \in A$ and $r_i, s_j \in R$, it follows by (4) that

$$(6)\qquad \delta[r_1,\cdots,r_{n^2};s_1,\cdots]a = \sum_i \pm\,\delta[r_1,\cdots,r_{i-1},a,r_{i+1},\cdots]r_i \in (A \cap C)R,$$

from which (1) follows immediately.

To prove (2), let $\delta = \delta[r_1,\cdots,r_{n^2};s_1,\cdots] \in \Delta$; then it follows by (4) that, for every $r \in R$, $\delta r = \Sigma_{i=1}^m c_i r_i$, where $c_i = \delta[r_1,\cdots,r_{i-1},r,\cdots] \in C$. Hence the ring of quotients R_δ is generated over C_δ by $r_1,\cdots,r_{n^2}$. These elements are also C_δ-independent: for let $\Sigma c_j r_j = 0$ in R_δ, i.e. $\Sigma \delta^k c_j r_j = 0$ for some power δ^k. Hence,

$$\begin{aligned} 0 &= \delta\left[r_1,\cdots,r_{i-1},\sum \delta^k c_j r_j, r_{i+1},\cdots,r_m;s_1,\cdots,s_t\right] \\ &= \sum_j \delta^k c_j \delta[r_1,\cdots,r_{i-1},r_j;r_{i+1},\cdots,r_m;s_1,\cdots,s_t] = \delta^{k+1}c_i \end{aligned}$$

since, for all $j \neq i$, $\delta[r_1,\cdots,r_{i-1},r_j,\cdots] = 0$ as two of its entries are equal, and for $i = j$, $\delta[r_1,\cdots,r_{i-1},r_i,\cdots] = \delta$. The last relation means that for all i, $c_i = 0$ in R_δ.

Q.E.D.

Part (3) follows now easily, for let $\delta = \delta[r_1, \cdots, r_{n^2}; s_1, \cdots] \neq 0$ and choose $a_i = \delta r_i$. It follows by (4) that $\delta r = \sum_{i=1}^{m} c_i r_i$, $c_i \in C$ and hence $a_i a_j = \delta(\delta r_i r_j) = \delta \sum c_{ijk} r_k = \sum c_{ijk} a_k$, i.e. $A = \sum C a_i$ is a subalgebra of R.Furthermore, the previous relation shows that $\delta^2 R \subseteq \delta \sum C r_i = \sum C a_i = A$. Finally, the $\{a_i\}$ and the $\{r_i\}$ are C-independent, for if $\sum c_i r_i = 0$ then, as in the previous proof, it follows that $\delta^k c_i = 0$, and thus $c_i = 0$. Clearly R is embeddable in the free C-module $\sum C(r_i/\delta)$ in R_δ.

This extends a result which was proved for prime rings by Formanek ([5]).

4. Application II: M. Artin theorem

We apply the previous results combined with an idea of Rowen [8] to obtain a straightforward proof of the famous theorem of M. Artin ([3]), which we prove in a more general but in a slightly different form:

THEOREM 7. *Let R be an algebra with a unit; then R is an Azumaya algebra of rank n^2 if and only if R satisfies the identity $d_{n^2-1}[x; y] = 0$ and*

(A) *R has an alternating central identity $\delta[x_1, \cdots, x_{n^2}; y_1, \cdots, y_m]$ which does not vanish in every quotient R/M, M a maximal ideal in R.*

A condition equivalent to (A) is:

(A′) *R has a regular central identity of the type of Corollary 5 and for every maximal ideal R/M is central simple of dimension n^2 over its center (or equivalently $d_{n^2}[x; y] \neq 0$ in R/M).*

PROOF. First we assume (A) and we prove that for every prime ideal $\mathfrak{P}$ in the center C of R there exists a prime ideal P in R such that $P \cap C \subseteq \mathfrak{P}$ and $P \supseteq \mathfrak{P}$ (hence $P \cap C = \mathfrak{P}$).

Indeed, let P be an ideal in R maximal with respect to the property $P \cap C \subseteq \mathfrak{P}$. Then P is a prime ideal, for if $AB \subseteq P$ with A, B two ideals containing P, then $(A \cap C)(B \cap C) \subseteq AB \cap C \subseteq \mathfrak{P}$. Since $\mathfrak{P}$ is prime, say $A \cap C \subseteq \mathfrak{P}$, then by the maximality of P it follows that $A \subseteq P$ as required. The alternating central polynomial $\delta[x; y]$ does not vanish on R/P since it does not vanish on some R/M for some maximal $M \supseteq P$, so let $\delta_0 = \delta[a_1, \cdots, b_m] \notin P$. Assume that $P \cap C \neq \mathfrak{P}$ and let $\alpha \in \mathfrak{P}$, $\alpha \notin P$; then since P is a prime ideal, $\alpha\delta_0 \notin P$ and, therefore, $(\alpha\delta_0 R + P) \cap C \not\subseteq \mathfrak{P}$ by the maximality of P. Consequently, $\beta = \alpha\delta_0 r + p$ for some central $\beta \notin \mathfrak{P}$ and $p \in P$. In the quotient ring $R/P = \bar{R}$ we have $\bar{\beta} = \bar{\alpha}\bar{\delta}_0\bar{r}$ and $\bar{r} \in \mathrm{Cent}(\bar{R})$; thus $\bar{\delta}_0\bar{r} = \delta[\bar{a}_1, \cdots, \bar{b}_m]\bar{r} = \delta[\bar{a}_1\bar{r}, \cdots, \bar{b}_m] = \bar{\delta}_1$, where $\delta_1 = \delta[a_1 r, \cdots, b_m] \in C$, and hence $\bar{\beta} = \bar{\alpha}\bar{\delta}_1$. It follows, therefore, that $\beta - \alpha\delta_1 \in P$ and, as β, $\alpha\delta_1 \in C$, $\beta - \alpha\delta_1 \in P \cap C \subseteq \mathfrak{P}$; but $\alpha \in \mathfrak{P}$ and so $\beta \in \mathfrak{P}$, which is a contradiction. Hence $\mathfrak{P} \subseteq P \cap C \subseteq P$.

Now let $\mathfrak{M}$ be a maximal ideal in C; then by our previous result there exists a maximal ideal M in R such that $M \supseteq \mathfrak{M}$. By assumption $\delta[x;y]$ does not vanish on R/M so we get $\delta = \delta[a;b] \not\equiv 0 \bmod M$ and hence $\delta \not\in \mathfrak{M}$. It follows now by (2) of Theorem 6 that R_δ is a free C_δ-module of rank n^2; this proves by Bourbaki [2] (II, theor. 2, p. 141) that R is C-projective of rank n^2. Finally, by (1) of Theorem 6, $\delta M \subseteq (M \cap C)R \subseteq \mathfrak{M}R$. Since $\mathfrak{M}$ is maximal and $\delta \not\in \mathfrak{M}$ we have $\alpha\delta \equiv 1 \bmod \mathfrak{M}$ for $\alpha \in C$ and therefore, $M \subseteq \alpha\delta M + \mathfrak{M}R \subseteq \mathfrak{M}R \subseteq M$. Consequently, $M = \mathfrak{M}R$ and thus $R/\mathfrak{M}R = R/M$ is by assumption a central simple algebra of dim n^2 over its center. Thus, by Bourbaki [2] (def. 14 of II, §5, p. 180), R is an Azumaya algebra of rank n^2. Note, that a simple consequence of this result is $\Delta = C$, since $\Delta \not\subseteq \mathfrak{M}$ for every maximal ideal $\mathfrak{M}$.

The converse: if R is an Azumaya algebra of rank n^2 then R satisfies all identities of $M_n(Z)$; in particular it will satisfy $d_{n^2+1}[x;y] = 0$ and, by Theorem 3, it has an alternating central identity which does not vanish in every $M_n(\bar{K})$ and hence also in every R/M.

Finally, (A′) is equivalent to (A): for if R satisfies (A′) then by Corollary 5, R has an alternating central identity $\delta[x;y]$ which does not vanish for all $M_n(\bar{K})$. Now, each R/M is a prime algebra over some $\bar{K} = K + M/M$ and R/M is by assumption central simple of dimension n^2 over its center; hence it follows by the proof of Theorem 3 that $\delta[x;y]$ will not vanish on R/M. Conversely if (A) holds, then since R/M satisfies $d_{n^2+1}[x;y] = 0$ it follows that $\mathrm{pid}(R/M) \leq n$. On the other hand, the fact that $\delta[x_1, \cdots, x_{n^2}; y_1, \cdots] \neq 0$ on R/M and that $\delta[x;y]$ vanishes if the n^2 x_i are linearly dependent over the center of R/M proves that $\mathrm{pid}(R/M) = n$, i.e. R/M is central simple of dimension n^2 over its center. Furthermore, the existence of such an identity follows from the fact that R satisfies all identities of $M_n(Z)$. Note that in the proof one needs only that the regular identity of R will not necessarily hold for all $M_n(\bar{K})$ as in Corollary 5, but for those $\bar{K} = K/K \cap M \cong K + M/M$, M maximal ideals of R.

An immediate consequence of the fact that $\Delta = C$ for an algebra satisfying Theorem 7 is that:

COROLLARY 7. *If R satisfies Theorem 7 (i.e. an Azumaya algebra of rank n^2) then $A = (A \cap C)R$ for every ideal A in R. Generally, for arbitrary rings R which satisfy Theorem 6, if $A \cap C + \Delta = C$ then $(A \cap C)R = A$.*

Indeed, by (1) of Theorem 6 it follows that $A = CA = \subseteq \Delta A + (A \cap C)A \subseteq (A \cap C)R \subseteq A$. For Azumaya algebras $\Delta = C$, and so this condition holds for all ideals A.

5. Application III: noetherian PI-rings

We again apply the previous result, in particular (4), to prove that under certain restrictions the ACC condition on two-sided ideals imply that the ring is right and left noetherian.

We consider ring R of the type described in Theorem 6, that is a ring R which satisfies $d_{m+1}[x;y]=0$ and has an alternative central identity $\delta[x_1,\cdots,x_m;y_1,\cdots,y_t]=\delta[x;y]$, and let Δ be the ideal in C generated by the values $\{\delta[a;b]\}$. Then:

THEOREM 8. *If R is a ring with this property and satisfies the ascending chain condition (ACC) on two sided ideals, the $R/(0:\Delta)$ is right and left noetherian, where*

$$(0:\Delta)=\{r\in R, R\Delta r=0\}.$$

In particular if R is a prime PI-ring, it satisfies the requirement of this theorem, and furthermore $\Delta(\neq 0)$ contains regular elements and so $(0:\Delta)=0$. Hence,

COROLLARY 9. (Cauchon [9]). *A prime* PI-*ring with the* ACC *condition on two sided ideals is noetherian.*

This was the basic result of Cauchon from which it follows that the corollary holds also for semi-prime rings.

PROOF. Since R satisfies the ACC on two-sided ideals, it follows that $R\Delta$ is finitely generated, i.e. $R\Delta=\Sigma R\delta_i$, where $\delta_i=\delta[a_{i1},\cdots,a_{im},b_{i1},\cdots]$.

Let ${}_RL$ be a left ideal in R, and the theorem will follow by showing that L is a finitely generated ideal mod$(0:\Delta)$:

For every $x\in L$, it follows by (4) that $\delta_i x=\Sigma_{j=1}^{m}c_{ij}(x)a_{ij}$ where $c_{ij}(x)\in C$. Consider the set of all vectors $c(x)=(c_{ij}(x))$ as elements of the direct sum $\bigoplus R=R^{(tm)}$ of copies of R. Since R is noetherian as an R-R module, so is also $R^{(tm)}$; hence, the module $N_L=\Sigma_{x\in L}Rc(x)$, which is a two-sided R submodule of $R^{(tm)}$, is finitely generated. Consequently, there are $u_1,\cdots,u_k\in L$ such that $N_L=\Sigma_{i=1}^{k}Rc(u_i)$.

Thus for every $x\in L$: $c(x)=\Sigma a_\rho c(u_\rho)$, $a_\rho\in R$ which means that $c_{ij}(x)=\Sigma_\rho a_\rho c_{ij}(u_\rho)$. It follows, therefore, by the definition of $(c_{ij}(x))$, that:

$$\delta_i x=\sum_j c_{ij}(x)a_{ij}=\sum_j\sum_\rho a_\rho c_{ij}(u_\rho)a_{ij}=\sum_\rho a_\rho\sum_j c_{ij}(u_\rho)a_{ij}=\sum_\rho \delta_i a_\rho u_\rho .$$

Hence, $\delta_i(x-\Sigma_\rho a_\rho u_\rho)=0$ which yields $R\Delta(x-\Sigma a_\rho u_\rho)=0$. This proves that $L\equiv\Sigma_\rho Ru_\rho \mod(0:\Delta)$. Q.E.D.

We note that the same proof with a slight modification yields:

THEOREM 10. *Let R satisfy $d_{m+1}[x;y]=0$ and have an alternative central identity $\delta[x;y]$. If its center C is noetherian then $R/(0:\Delta)$ is a noetherian C-module; and hence R is also right and left noetherian.*

Indeed replace $R\Delta$ by Δ, which is a C-ideal, and $R^{(m)}$ by $C^{(m)} = \bigoplus C$, and the same arguments will yield the elements $a_\rho \in C$, and the rest is similar.

We conclude with two remarks on the ideal $(0:\Delta)$ and on the existence of nonprime rings with an identity $\delta[x;y]$ for which $(0:\Delta)=0$.

THEOREM 11. *Let R be a semi-prime ring with* $\mathrm{pid}\,R = n$, *and* $N_n = \bigcap\{P$; *prime and* $\mathrm{pid}(R/P) = n\}$. *Then for every alternating central identity $\delta[x;y]$ of Corollary 5 (which exists!) we have* $(0:\Delta) = N_n$.

Indeed, first note that $\delta[x;y] = \delta[x_1,\cdots,x_{n^2};y_1,\cdots]=0$ in every quotient R/P for which $\mathrm{pid}\,R/P<n$, since $\delta[x;y]=0$ vanishes if the x_i and dependent over the center of R/P, which is always the case if $\mathrm{pid}\,R/P<n$. Hence, $\Delta N_n \subseteq P$ for every prime ideal in R since either $\Delta \subseteq P$ or $N_n \subseteq P$, and so $\Delta N_n = \cap P = 0$ in the semi-prime ring R. Thus $N_n \subseteq (0:\Delta)$.

To prove the converse, let $a \notin N_n$; then $a \notin P$ for some prime P with $\mathrm{pid}\,R/P = n$. But $\delta[x;y]\not\equiv 0 \mod P$ by the proof of Corollary 5, and so $\delta[u;v]a \not\equiv 0 \mod P$ for some $u_i, v_j \in R$. Thus $a \notin (0:\Delta)$ and, therefore, $(0:\Delta)\subseteq N_n$. Q.E.D.

REMARK. It is worth pointing out that, with the exception of Application II, (Section 4) R is not assumed to contain a unit.

The last theorem enables us to introduce an induction process in handling semi-prime rings:

LEMMA 12. *Let R be a semi-prime ring with* $\mathrm{pid}\,R = n$ *with center C, then the subring $R' = C + N_n$ is a semi-prime ring with the same center and* $\mathrm{pid}\,R' < n$.

It is easily verified that R' is semi-prime. Let $N_n' = \cap P$ where P ranges over all primes for which $\mathrm{pid}\,R/P<n$; then $N_n' \cap N_n = 0$ in the semi-prime ring R. Hence, $N_n \cong (N_n + N_n')/N_n' \subseteq R/N_n'$ and so N_n will also satisfy $d_{m+1}[x;y]=0$ with $m < n^2$. Now R' is a central extension of N_n and therefore will also satisfy the same identity, which yields $\mathrm{pid}\,R'<n$.

Also, if $u \in N_n \cap \mathrm{Cent}(R')$, then for every $r \in R$, $a \in N_n$, $0 = u(ar)-(ar)u = a(ur-ru)$, and so $ur - ru \in N_n \cap (0:N_n) = 0$. Thus, $u \in \mathrm{Cent}\,R = C$.

This induction process clearly yields Formanek's result:

COROLLARY 13. (Formanek [5]). *A semi-prime* PI-*ring* R *with a noetherian center C is C-noetherian and hence noetherian.*

The ring R/N_n is C-noetherian by Theorems 10 and 11. The C-module N_n is a submodule of R', on which by the previous lemma we can use an induction on the pid of the ring. So N_n is also a noetherian C-module, and the proof follows now from the exact sequence $0 \to N_n \to R \to R/N_n \to 0$.

REFERENCES

1. S. A. Amitsur, *Polynomial identities and Azumaya algebras*, J. Algebra **27** (1973), 117–125.
2. N. Bourbaki, *Algèbre commutative*, Ch. 2, Hermann, Paris, 1961.
3. M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Algebra **11** (1969), 532–563.
4. E. Formanek, *Central polynomials for matrix rings*, J. Algebra **23** (1972),129–132.
5. E. Formanek, *Noetherian* PI-*rings* (to appear).
6. C. Procesi, *On a theorem of M. Artin*, J. Algebra **22** (1972), 306–309.
7. Ju. P. Razmyslov, *On the Kaplansky problem* (Russian), Izv. Akad. Nauk. SSSR, Ser. Mat. **37** (1973), 483–501.
8. L. H. Rowen, *On rings with central polynomials*, J. Algebra **31** (1974), 393–426.
9. G. Cauchon, *Anneaux semi-premiers, noetheriens, à identités polynômiales* (to appear).

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